B.Sc. Physics Part-III 21-09-2020

Paper-V, Group-A

Series solutions of differential equations:

→ Many differential equations cannot be solved in terms of elementary functions. These equations can be solved using numerical methods and in many cases we can find solutions of these equations in terms of infinite convergent series.

We consider here a simple differential equation and obtain its solution using method of series solution.

$$\frac{dy}{dx} = 2xy \qquad \text{--- (1)}$$

Note that equation (1) can easily be solved by elementary method also but we use here method of series solution.

We assume that eqn. (1) has the solution of the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$

or $$y = \sum_{m=0}^{\infty} a_m x^m \qquad \text{--- (2)}$$

In eqn. (2) the coefficients a's are to be determined.

From eq. (2); $\frac{dy}{dx} = a_1 + 2a_2 x + \cdots + m a_m x^{m-1}$

or $$\frac{dy}{dx} = \sum_{m=1}^{\infty} m a_m x^{m-1} \qquad \text{--- (3)}$$

We substitute eqns (2) and (3) in eqn. (1)

$$\sum_{m=1}^{\infty} m a_m x^{m-1} = 2x \sum_{m=0}^{\infty} a_m x^m \quad \text{--- (4)}$$

Or

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots + m a_m x^{m-1} = 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 + \dots + 2a_m x^{m+1}$$

The ~~two coefficients~~ series should be identical. which means that the coefficients of the corresponding powers of x must be equal from eq. (4)

$$a_1 = 0$$

$$a_2 = a_0$$

$$3a_3 = 2a_1 \Rightarrow a_3 = \frac{2}{3} a_1 = 0$$

$$4a_4 = 2a_3 \Rightarrow a_4 = \frac{2}{4} a_3 = 0$$

$$\vdots$$

$$m a_m = 2a_{m-2} \Rightarrow a_m = \frac{2}{m} a_{m-2}$$

$$a_m = 0, \text{ if } \cancel{m \text{ is odd}}\ m \text{ is odd.}$$

$$\frac{2}{m} a_{m-2}, \text{ for even } m$$

Next, we put $m = 2n$ [because odd terms are zero and we need to bother about only even terms]

$$\therefore a_{2n} = \frac{2}{2n} a_{2n-2} = \frac{1}{n} a_{2n-2}$$

$$\text{or} \quad a_{2n} = \frac{1}{n}\left(\frac{1}{n-1} a_{2n-4}\right) = \frac{1}{n(n-1)} a_{2n-4}$$

$$= \frac{1}{n(n-1)} \frac{1}{(n-2)} a_{2n-6} = \dots \frac{1}{n!} a_0$$

Therefore, the solution of eqn. (1) is given

$$y = a_0 + a_0 x^2 + \frac{1}{2!} a_0 x^4 + \dots + \frac{1}{n!} a_0 x^{2n}$$

$$\text{or} \quad \boxed{y = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}}$$